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2009 J. Phys. A: Math. Theor. 42 442004

(http://iopscience.iop.org/1751-8121/42/44/442004)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 442004 (5pp) doi:10.1088/1751-8113/42/44/442004

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The short-pulse equation and associated constraints

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Received 31 August 2009, in final form 23 September 2009
Published 9 October 2009
Online at stacks.iop.org/JPhysA/42/442004

Abstract
The short-pulse equation (SPE) is considered as an initial boundary value problem. It is found that the solutions of the SPE must satisfy an integral relation otherwise the temporal derivative exhibits discontinuities. This integral relation is not necessary for a solution to exist. An infinite number of such constraints can be dynamically generated by the evolution equation.

PACS numbers: 02.30.Ik, 02.30.Jr

The standard model for describing the propagation of a pulse-shaped complex field envelope in nonlinear dispersive media is the nonlinear Schrödinger (NLS) equation. In the context of nonlinear optics, the main assumption made when deriving the NLS equation from Maxwell's equations is that the pulse width is large as compared to the period of the carrier frequency. When this assumption is no longer valid, i.e., for pulse duration of the order of a few cycles of the carrier, the evolution of such 'short pulses' is better described by the so-called short-pulse equation (SPE) [1].

The SPE can be expressed in the following dimensionless form,

$$u_{xt} = u + \tfrac{1}{6}(u^3)_{xx}, \tag{1}$$

where subscripts denote partial derivatives. The SPE forms an initial boundary problem when accompanied by the initial data

$$u(0, x) = u_0,$$

and sufficiently fast decaying boundary conditions $u(t, \pm\infty) = 0$. Much like the NLS equation, the SPE is integrable [2] and exhibits soliton solutions in the form of loop-solitons [3]. However, when it is formed as an evolution equation certain conditions must apply otherwise, as shown below, the temporal derivative exhibits discontinuities.

Despite the fact that the equation is integrable via the inverse scattering transform [4], there are certain subtleties that need to be clarified. Direct integration of equation (1) introduces the operation

$$\partial_x^{-1} u(t, x) = \int_{-\infty}^{x} u(t, x')\,\mathrm{d}x'.$$

Clearly as x approaches $-\infty$, $\partial_x^{-1} u = 0$, consistent with rapidly decaying data. However, as x approaches $+\infty$, for u and its time and space derivatives to decay, a constraint seems to be necessary (see the discussion below), namely

$$\int_{-\infty}^{+\infty} u(t,x)\,\mathrm{d}x = 0. \tag{2}$$

Indeed, writing the SPE in an evolution-type form we have

$$u_t = \partial_x^{-1} u + \frac{1}{6}(u^3)_x = \int_{-\infty}^{x} u\,\mathrm{d}x' + \frac{1}{6}(u^3)_x$$

and imposing the boundary condition as $x \to +\infty$, one results to equation (2).

In fact, this constraint induces further constraints obtained by successively taking the time derivative of the integral and using equation (1). For example, the next constraint is given by

$$\int_{-\infty}^{+\infty} \partial_x^{-1} u\,\mathrm{d}x = 0. \tag{3}$$

However, equations (2) and (3), along with the rest of the family of infinite constraints generated as above, are not generically true. One might surmise that constraints are required at all times for a solution to exist. However, as discussed below, this is not the case. Extra constraints on the initial data are not necessary, but the solution suffers from a temporal discontinuity. For smooth initial data not satisfying equation (2), $u_t(t,x)$ has at $t = 0$ different left and right limits and the rest of the family of constraints cannot be generated dynamically at that point. The same issues arise in the context of the Kadomtsev–Petviashvili (KP) equations and were studied in [5, 6].

Our analysis starts by taking the Fourier transform (FT) of equation (1),

$$\mathrm{i}k\hat{u}_t = \hat{u} - \frac{k^2}{6}\widehat{u^3}, \tag{4}$$

where the FT pair is defined as

$$\hat{u}(t,k) = \mathcal{F}\{u(t,x)\} = \int_{-\infty}^{+\infty} u(t,x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$$

$$u(t,x) = \mathcal{F}^{-1}\{\hat{u}(t,k)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}(t,k)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k.$$

Define $\hat{U} = \widehat{u^3}$ and write equation (4) in the form of a first-order differential equation in t,

$$\hat{u}_t - \frac{1}{\mathrm{i}k}\hat{u} = \frac{\mathrm{i}k}{6}\hat{U} \tag{5}$$

which can be readily integrated with the use of integrating factors to give

$$\hat{u}(t,k) = \hat{u}_0\,\mathrm{e}^{t/\mathrm{i}k} + \frac{\mathrm{i}k}{6}\mathrm{e}^{t/\mathrm{i}k}\int_0^t \hat{U}\,\mathrm{e}^{-\tau/\mathrm{i}k}\,\mathrm{d}\tau, \tag{6}$$

where $\hat{u}_0 = \hat{u}_0(k) = \mathcal{F}\{u_0(x)\}$, is the FT of the initial data. Using equation (6) we calculate the temporal derivative to be

$$\hat{u}_t(t,k) = \frac{1}{\mathrm{i}k}\hat{u}_0\,\mathrm{e}^{t/\mathrm{i}k} + \frac{\mathrm{i}k}{6}\hat{U} + \frac{1}{6}\int_0^t \hat{U}\,\mathrm{e}^{(t-\tau)/\mathrm{i}k}\,\mathrm{d}\tau. \tag{7}$$

Clearly, from equation (5), as k tends to zero, we should demand that so will $\hat{u}$. This translates into [7]

$$\hat{u}(t,0) = 0 \quad \Leftrightarrow \quad \int_{-\infty}^{+\infty} u(t,x)\,\mathrm{d}x = 0.$$

However, as $t \to \pm 0$, we have that $\hat{u}(0, x) = \hat{u}_0$ from equation (6), and

$$\hat{u}_t = \frac{1}{\mathrm{i}k - \mathrm{sign}(t)0}\hat{u}_0 + \frac{\mathrm{i}k}{6}\hat{U}.$$

This is because the function $\exp(t/\mathrm{i}k)$ defines a distribution, depending continuously on t, in the Schwartz space of the variable k [8] with

$$\frac{\partial}{\partial t}\,\mathrm{e}^{t/\mathrm{i}k} = \frac{1}{\mathrm{i}k - \mathrm{sign}(t)0}\,\mathrm{e}^{\mathrm{i}t/k}, \quad t = 0$$

and

$$\frac{\partial}{\partial t}\,\mathrm{e}^{t/\mathrm{i}k} = \frac{1}{\mathrm{i}k}\,\mathrm{e}^{t/\mathrm{i}k}, \quad t \neq 0.$$

This suggests that although there is no discontinuity in the solution, there is one in the derivative. Indeed, taking the inverse FT of equation (7), at $t \to \pm 0$, we have

$$u_t(t \to \pm 0, x) = \frac{1}{2\pi}\lim_{t\to\pm 0}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}k}\hat{u}_0\,\mathrm{e}^{t/\mathrm{i}k}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k + \frac{1}{6}(u^3)_x(t \to \pm 0, x).$$

The nonlinear term is straightforward to handle so we focus on the linear part,

$$\begin{aligned} I(x) &= \frac{1}{2\pi}\lim_{t\to\pm 0}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}k}\hat{u}_0\,\mathrm{e}^{t/\mathrm{i}k}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \\ &= \frac{1}{2\pi}\lim_{t\to\pm 0}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}k}[\hat{u}_0\,\mathrm{e}^{-\mathrm{i}kx} + \hat{u}_0(0) - \hat{u}_0(0)]\,\mathrm{e}^{t/\mathrm{i}k}\,\mathrm{d}k \\ &= \frac{1}{2\pi}\lim_{t\to\pm 0}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}k}[\hat{u}_0\,\mathrm{e}^{-\mathrm{i}kx} - \hat{u}_0(0)]\,\mathrm{e}^{t/\mathrm{i}k}\,\mathrm{d}k \\ &+ \frac{1}{2\pi}\lim_{t\to\pm 0}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}k}\,\hat{u}_0(0)\,\mathrm{e}^{t/\mathrm{i}k}\,\mathrm{d}k. \end{aligned} \tag{8}$$

Using the property

$$\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}k}\,\mathrm{e}^{t/\mathrm{i}k}\,\mathrm{d}k = -\pi\,\mathrm{sign}(t)$$

the second integral of equation (8) is reduced to $-\hat{u}_0(0)\pi\,\mathrm{sign}(t)/2$. Furthermore, we write

$$\hat{u}_0(0) = \int_{-\infty}^{+\infty}\delta(k)\hat{u}_0(k)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k$$

so that finally

$$\begin{aligned} I(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\mathrm{P}\left(\frac{1}{\mathrm{i}k}\right) - \pi\,\mathrm{sign}(t)\delta(k)\right]\hat{u}_0(k)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\hat{u}_0(k)}{\mathrm{i}k - 0\,\mathrm{sign}(t)}\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \\ &= \int_{\mathrm{sign}(t)\infty}^{x} u_0(x')\,\mathrm{d}x' \end{aligned}$$

where

$$P\left(\frac{1}{x}\right)(u) = \lim_{\epsilon\to 0^+}\int_{|x|>\epsilon}\frac{u(x)}{x}\,\mathrm{d}x$$

denotes the principal value for smooth functions with compact support on the real line, and $0\,\mathrm{sign}(t)$ (also denoted as $0^{\pm}$ in the literature) denotes infinitesimal values around 0, (−) left and (+) right.

Thus, at $t = 0$, equation (7) translates into physical space as

$$u_t = \int_{\mathrm{sign}(t)\infty}^{x} u_0(x')\,\mathrm{d}x' + \frac{1}{6}(u_0)_x.$$

As also mentioned in [6, 7], the operator

$$\partial_x^{-1} = \int_{\mathrm{sign}(t)\infty}^{x} \mathrm{d}x'$$

and its relative average

$$\partial_x^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} \mathrm{d}x' + \int_{x}^{\infty} \mathrm{d}x'\right)$$

are equivalent, meaning that one can choose either one of them. If $t \neq 0$ we have that $\int_{-\infty}^{+\infty} u\,\mathrm{d}x = 0$, hence both choices are valid. At $t = 0$ there is a discontinuity in the temporal derivative.

For the evolution of the SPE, equation (2) is not preserved in time and as such leads to the infinite number of further constraints. Indeed, if equation (2) holds, then an infinite number of constraints, dynamically generated using the SPE, hold during the evolution. Within the physical framework of the SPE these constraints are neither 'natural' nor necessary. Solutions of the SPE can exist without satisfying this condition, the most prominent example being the loop-soliton [3]. This solution, however, in addition to the possible temporal discontinuities, suffers from discontinuities in its spatial derivatives, $u_x(t, x)$, and extra care may be needed when the above formalism is applied.

We conclude with a note on the so-called regularized SPE (RSPE) model, recently derived in [9]. The RSPE has been derived by including a regularization term, based on the next term in the expansion of the dielectric's susceptibility. In that case, pulses (of the real component of the electric field) are described by

$$u_{xt} = u + \tfrac{1}{6}(u^3)_{xx} + \beta u_{xxxx},$$

where β is a small parameter. Without the regularization term, βu_{xxxx}, i.e., in the case of the SPE –cf equation (1)–, traveling pulses in the class of piecewise smooth functions with one discontinuity do not exist. However, when the regularization term is added, and for a particular parameter regime, the RSPE supports smooth traveling waves which have structure similar to solitary waves of the modified KdV equation [9]. The regularization term does not alter the analysis for the SPE. Indeed, in the Fourier domain the term is written as $\mathcal{F}\{\beta u_{xxxx}\} = \beta k^4 \hat{u}$ and thus when dividing with ik from the left-hand side the resulting power of k is continuous at $k = 0$. The linear part of the RSPE, much like the linear part of the KP-I equation [5, 8], deserves more study and the analysis will be presented in a future communication.

Acknowledgments

I would like to thank Mark J Ablowitz for bringing the KP analysis to my attention and Barbara Prinari, Dimitri J Frantzeskakis and Panayotis G Kevrekidis for many useful discussions.

References

[1] Schäfer T and Wayne C E 2004 Propagation of ultra-short optical pulses in cubic nonlinear media *Physica* D **196** 90–105

[2] Sakovich A and Sakovich S 2005 The short pulse equation is integrable *J. Phys. Soc. Japan* **74** 239–41

[3] Sakovich A and Sakovich S 2006 Solitary wave solutions of the short pulse equation *J. Phys. A: Math. Gen.* **39** L361–7

[4] Victor K K, Thomas B B and Kofane T C 2007 On exact solutions of the Schäfer–Wayne short pulse equation: WKI eigenvalue problem *J. Phys. A: Math. Theor.* **40** 5585–96

[5] Ablowitz M J and Wang X-P 1997 Initial time layers and Kadomtsev–Petviashvili-type equations *Stud. Appl. Math.* **98** 121–37

[6] Ablowitz M J and Villarroel J 1991 On the Kadomtsev–Petviashvili equation and associated constraints *Stud. Appl. Math.* **85** 195–213

[7] Pelinovsky D and Sakovich A 2008 Global well-posedness of the short-pulse and sine-Gordon equations in energy space arXiv:0809.5052v1

[8] Boiti M, Pempinelli F and Pogrebkov A 1994 Solutions of the KPI equation with smooth initial data *Inverse Problems* **10** 505–19

[9] Costanzino N, Manukian V and Jones C K R T 2008 Solitary waves of the regularized short pulse and Ostrovsky equations arXiv:0809.3294v1